SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

V. THE ISOPERIMETRIC PROBLEM IN PARAMETRIC FORM*

BY
E. J. McSHANE

1. **First existence theorem.** We continue the notation of preceding papers with the trifling change that z will denote $(z^1, \cdots, z^q)$ instead of $(z^0, \cdots, z^q)$ as heretofore, and a similar change for z'. The class of all rectifiable curves joining two fixed points z_1, z_2 (not necessarily distinct) will be denoted by K. The functions $F(z, z')$ and $G(z, z')$ are defined for all points $z = (z^1, \cdots, z^q)$ in a closed point set S and all z'. They are positively homogeneous of degree 1 in z', are continuous for all z in S and all z', and possess partial derivatives of first and second orders continuous except at $z' = 0$. We write

$$\mathcal{J}(C) = \int_C F(z, \dot{z})dt, \qquad \mathcal{G}(C) = \int_C G(z, \dot{z})dt.$$

A hypothesis which we shall henceforth impose on our integrals is the following:

(1.1) *To each pair of numbers l, m there corresponds a number L such that if C is in K and $|\mathcal{G}(C)| \leqq l$, $\mathcal{J}(C) \leqq m$, then $\mathcal{L}(C) \leqq L$.*

There are well known conditions which ensure this. For instance, if there are numbers $a \geqq 0$ and b such that $aF + bG$ is positive definite and the set S is bounded, then (1.1) is satisfied. Or, S being unbounded, if there exist numbers $a \geqq 0$ and b for which

$$aF(z, z') + bG(z, z') \geqq k\,|\,z'\,|\,(1 + |\,z\,|)^{-1},$$

$k > 0$, then (1.1) holds.

The proofs in note IV give us, with hardly any modification, a proof of the following theorem:

---

* Presented to the Society, April 15, 1938; received by the editors October 29, 1937 and, in revised form, February 16, 1938.

The preceding notes in this series have already appeared in these Transactions as follows: I, vol. 44 (1938), pp. 429–438; II, vol. 44 (1938), pp. 439–453; III, vol. 45 (1939), pp. 151–171; IV, vol. 45 (1939), pp. 173–196. They will be referred to by their numbers.

THEOREM 1. *Let the set S consist of the entire z-space, and let* (1.1) *hold. Let $\mathcal{G}(C)$ be quasi-regular normal. Suppose further that for each z, each approach set A at z contains only a finite number of unit vectors $p_1, \cdots, p_k$ which can be so ordered that $\Omega_H(z, p_i, p_j) < 0$ if $i < j$, where $H(z, r) = H(z, r; \lambda(z, A))$. Then for every number l, either the class $K[\mathcal{G}=l]$ is empty or it contains a curve which minimizes $\mathcal{J}(C)$ on the class $K[\mathcal{G}=l]$.*

The proofs in IV leading up to equations (8.2) and (8.3) of IV were so designed as to apply to problems in parametric form as well as to those in non-parametric form. Hence these equations, which are $\phi'(t) = \phi_0'(t)$ and $\gamma'(t) = \gamma_0'(t)$ for all t in E (that is, for almost all t in $[0, 1]$) are here valid. But for problems in parametric form the functions $\phi, \phi_0, \gamma, \gamma_0$ are Lipschitzian; so this implies that $\phi(1) = \phi_0(1)$ and $\gamma(1) = \gamma_0(1)$. As remarked at the end of §4 of IV, these equations imply the conclusion of the theorem.

2. **Statement of the principal existence theorem.** Before stating the next problem we introduce several definitions:

(2.1) *The point z_0 of S is an ordinary point if the following conditions hold:*

(a) *It is interior to S.*

(b) *$G(z, r)$ is quasi-regular normal (either positive or negative) at z_0.*

(c) *Each approach set A at z_0 contains only a finite number of unit vectors $p_1, \cdots, p_k$, and these can be so ordered that $\Omega_H(z_0, p_i, p_j) < 0$ if $i < j$; here $H(z, r) = F(z, r) - \lambda(z_0, A)G(z, r)$.*

Theorem 1 required that S be the whole space and that every point z_0 be ordinary. Subject to further hypotheses, our next theorem will permit S to be a closed subset of the space and will allow S to contain singular points (that is, points which are not ordinary points).

One of our hypotheses will be the following:

(2.2) *For each z_0 in S there is a number θ such that $F(z, r) - \theta G(z, r)$ is p.q.r.*† *at z_0.*

We denote by $m(z_0)$, $M(z_0)$, respectively, the greatest lower bound and the least upper bound of all numbers θ for which $F(z, r) - \theta G(z, r)$ is p.q.r. at z_0. It is easy to see that $m(z) = -\infty$ if and only if $G(z, r)$ is p.q.r. at z_0. For if $G(z, r)$ is p.q.r. at z_0 and θ_0 is any number such that $F - \theta_0 G$ is p.q.r. at z_0, then $F(z, r) - \theta G(z, r) = [F(z, r) - \theta_0 G(z, r)] + (\theta_0 - \theta)G(z, r)$ is also p.q.r. for all $\theta < \theta_0$; so $m(z) = -\infty$. However, if G is not p.q.r. at z_0 there are orthogonal unit vectors p, u for which $u^\alpha G_{\alpha\beta}(z, p)u^\beta < 0$. Then $u^\alpha [F_{\alpha\beta}(z, p) - \theta G_{\alpha\beta}(z, p)]u^\beta$ is negative if θ is negative and numerically large; so $m(z_0)$ is finite. Likewise $M(z_0) = +\infty$ if and only if $G(z, r)$ is n.q.r. at z_0.

† Defined in (6.2) of III.

For each curve C: $z=z(t)$, $(t_1 \leqq t \leqq t_2)$, lying in S we define two sets of points t as follows:

(2.3) *$T_+(C)$ is the set of all points t such that $z(t)$ is a singular point and $G(z, r)$ is not n.q.r. at $z(t)$.*

(2.4) *$T_-(C)$ is the set of all points t such that $z(t)$ is a singular point and $G(z, r)$ is not p.q.r. at $z(t)$.*

These sets may overlap: if $z(t)$ is a singular point and $G(z, r)$ is neither p.q.r. nor n.q.r. at $z(t)$, then t belongs to both $T_+(C)$ and $T_-(C)$. Also they may vanish simultaneously; this happens whenever $G(z, r)$ is linear in the variables r^i at each singular point $z(t)$.

Our next definition is as follows:

(2.5) *If C: $z=z(t)$ is a curve lying in S, then if $T_+(C)$ is not empty, we define $M(C)$ to be the greatest lower bound of $M(z(t))$ for all t in $T_+(C)$; if $T_-(C)$ is not empty, we define $m(C)$ to be the least upper bound of $m(z(t))$ for all t in $T_-(C)$.*

If $M(C)$ is defined, it is not $+\infty$. For if $M(C)$ is defined, the set $T_+(C)$ is not empty. Let t_0 be a point in it. At $z(t_0)$ the function $G(z, r)$ is not n.q.r.; so $M(z(t_0))$ is not $+\infty$. Hence $M(C) \leqq M(z(t_0)) < \infty$. Likewise, if $m(C)$ is defined, it is not $-\infty$. It is interesting to observe that if $G(z(t), r)$ is positive regular for all t and if C contains singular points $z(t)$, then $M(C)$ is defined and finite, while $m(C)$ is undefined. Likewise, if $G(z(t), r)$ is negative regular for all t and if C contains singular points, then $m(C)$ is defined and finite, while $M(C)$ is undefined. We prove the first statement; the proof of the second is similar. The set $T_+(C)$ here consists of all singular points, and by hypothesis is not empty; so $M(C)$ is defined. As always, $M(C) < \infty$. The quadratic form $u^\alpha G_{\alpha\beta}(z(t), p)u^\beta$ is positive for all t and all pairs of orthogonal unit vectors u, p. Let ν be its greatest lower bound; then $\nu > 0$.

The quadratic form $u^\alpha F_{\alpha\beta}(z(t), p)u^\beta$ is bounded, say by N, in absolute value, for the same arguments. Then $F(z(t), r)+(N/\nu)G(z(t), r)$ is p.q.r. for all t; so $M(z(t)) \geqq -N/\nu$ for each t, and $M(C) \geqq -N/\nu > -\infty$. Therefore $M(C)$ is finite.

We now state our principal theorem.

THEOREM 2. *Let the following hypotheses be satisfied:*

(a) *S is closed.*

(b) *Hypotheses* (1.1) *and* (2.2) *hold.*

(c) *For every curve C of K, either $T_+(C)$ is empty or there exists*† *a curve $\Gamma^*: z=\zeta(\tau)$, $(0 \leqq \tau \leqq \epsilon)$, with the properties:*

† This implies that $M(C)$ is finite. For $M(C)$ is never $+\infty$, and if $M(C)$ were $-\infty$, inequalities (2.6) and (2.7) could not hold. Likewise, from (d) we conclude that $m(C)$ is undefined or is finite.

(i) *The length of* Γ^* *is not zero.*

(ii) $\zeta(0)$ *is on* C.

(iii) *For almost all* τ *in* $[0, \epsilon]$ *the inequalities*

$$(2.6)\quad G(\zeta(\tau), \dot{\zeta}(\tau)) + G(\zeta(\tau), -\dot{\zeta}(\tau)) > 0,$$

$$(2.7)\quad F(\zeta(\tau), \dot{\zeta}(\tau)) + F(\zeta(\tau), -\dot{\zeta}(\tau)) \leqq M(C)[G(\zeta(\tau), \dot{\zeta}(\tau)) + G(\zeta(\tau), -\dot{\zeta}(\tau))]$$

hold.

(d) *For every curve* C *of* K, *either* $T_-(C)$ *is empty or there exists a curve* $\Gamma^*: z=\zeta(\tau)$, $(0\leqq\tau\leqq\epsilon)$, *with the properties*:

(i) *The length of* Γ^* *is not zero.*

(ii) $\zeta(0)$ *is on* C.

(iii) *For almost all* τ *in* $[0, \epsilon]$ *the inequalities*

$$(2.8)\quad G(\zeta(\tau), \dot{\zeta}(\tau)) + G(\zeta(\tau), -\dot{\zeta}(\tau)) < 0,$$

$$(2.9)\quad F(\zeta(\tau), \dot{\zeta}(\tau)) + F(\zeta(\tau), -\dot{\zeta}(\tau)) \leqq m(C)[G(\zeta(\tau), \dot{\zeta}(\tau)) + G(\zeta(\tau), -\dot{\zeta}(\tau))]$$

hold.

Then for every l *the class* $K[\mathcal{G}=l]$ *either is empty or contains a curve* C *for which* $\mathcal{J}(C)$ *assumes its least value on* $K[\mathcal{G}=l]$.

Sections 3 to 5 will be devoted to the proof of this theorem, and throughout these sections the hypotheses of Theorem 2 will be assumed to hold.

3. **Construction of a minimizing sequence.** If the class $K[\mathcal{G}=l]$ is not empty, we denote by μ the greatest lower bound of $\mathcal{J}(C)$ on the class $K[\mathcal{G}=l]$. Also we define μ_0 to be the greatest lower bound of numbers m for which there exists a sequence $\{C_n\}$ of curves of K having

$$\lim_{n\to\infty} \mathcal{J}(C_n) = m, \qquad \lim_{n\to\infty} \mathcal{G}(C_n) = l.$$

Since μ is such a number m, we evidently have $\mu_0\leqq\mu$. If $\{h_n\}$ is a sequence of numbers greater than μ_0 and tending to μ_0, for each n there is a C_n^* such that

$$\mathcal{J}(C_n^*) < h_n, \qquad |\mathcal{G}(C_n^*) - l| < 1/n.$$

Hence

$$(3.1)\qquad \lim_{n\to\infty} \mathcal{J}(C_n^*) = \mu_0, \qquad \lim_{n\to\infty} \mathcal{G}(C_n^*) = l.$$

By (1.1), the C_n^* have bounded lengths, which ensures the finiteness of μ_0.

Suppose C_n^* has the Lipschitzian representation $z=z_n^*(t)$, $(0\leqq t\leqq 1)$. The set of values of t such that $z_n^*(t)$ has a distance greater than $1/2n$ from the boundary of S is open relative to $[0, 1]$ and so falls into a finite or denumerable aggregate of subintervals. For only a finite number of these, say

$[\alpha_1, \beta_1], \cdots, [\alpha_h, \beta_h]$, can the corresponding arc of C_n^* have length as great as $1/2n$; the others we disregard. Thus if t is not in any of the intervals $[\alpha_i, \beta_i]$, the point $z_n^*(t)$ has distance less than n^{-1} from the boundary of S.

Consider now one of the intervals $[\alpha_i, \beta_i]$. In the corresponding arc of C_n^* we inscribe a polygon $\Pi_{i,n}^*$ with the successive vertices

$$z_n^*(\alpha_i), z_n^*(t_1), \cdots, z_n^*(t_m), z_n^*(\beta_i), \qquad \alpha_i < t_1 < \cdots < t_m < \beta_i.$$

If the sides of this polygon are short enough, we will have

$$\left| \mathcal{J}(\Pi_{i,n}^*) - \int_{\alpha_i}^{\beta_i} F(z_n^*, \dot{z}_n^*)dt \right| < 1/hn, \tag{3.2}$$

and

$$\left| \mathcal{G}(\Pi_{i,n}^*) - \int_{\alpha_i}^{\beta_i} G(z_n^*, \dot{z}_n^*)dt \right| < 1/hn. \tag{3.3}$$

Now let $\Pi_{i,n}^*$ be the polygon having the same number of vertices as $\Pi_{i,n}^*$, joining $z_n^*(\alpha_i)$ to $z_n^*(\beta_i)$, having $\mathcal{G}(\Pi_{i,n}) = \mathcal{G}(\Pi_{i,n}^*)$, and minimizing $\mathcal{J}(C)$ in that class of polygons. Such a polygon exists, by Lemma 3 of IV. For this polygon we have

$$\mathcal{J}(\Pi_{i,n}) < \int_{\alpha_i}^{\beta_i} F(z_n^*, \dot{z}_n^*)dt + 1/hn, \tag{3.4}$$

$$\left| \mathcal{G}(\Pi_{i,n}) - \int_{\alpha_i}^{\beta_i} G(z_n^*, \dot{z}_n^*)dt \right| < 1/hn. \tag{3.5}$$

Now we define the curve C_n to be the curve C_n^* with the arcs corresponding to the intervals $[\alpha_i, \beta_i]$ replaced by the respective polygons $\Pi_{i,n}$. Since we can consider the functions z_n^* unaltered except on the intervals $[\alpha_i, \beta_i]$, we obtain from (3.4) and (3.5) the relations

$$\mathcal{J}(C_n) < \mathcal{J}(C_n^*) + 1/n, \tag{3.6}$$

$$| \mathcal{G}(C_n) - \mathcal{G}(C_n^*) | < 1/n. \tag{3.7}$$

Hence

$$\limsup_{n\to\infty} \mathcal{J}(C_n) \leqq \lim_{n\to\infty} \mathcal{J}(C_n^*) + 0 = \mu_0, \tag{3.8}$$

$$\lim_{n\to\infty} \mathcal{G}(C_n) = l. \tag{3.9}$$

But by the definition of μ_0 we cannot have $\liminf \mathcal{J}(C_n) < \mu_0$, in the presence of (3.9). So from this and (3.8) we conclude that

(3.10) $$\lim_{n\to\infty} \mathcal{J}(C_n) = \mu_0 \leqq \mu.$$

By (1.1), the curves C_n are of uniformly bounded lengths; so we can select a subsequence for which $\mathcal{L}(C_n)$ converges to a finite limit L:

(3.11) $$\lim_{n\to\infty} \mathcal{L}(C_n) = L < \infty.$$

Clearly we may suppose (since we can discard a finite number of the C_n) that we have

(3.12) $$\mathcal{L}(C_n) \leqq L + 1.$$

For this subsequence (3.9) and (3.10) still hold.

On each curve C_n we introduce as parameter $t = s/\mathcal{L}(C_n)$, where s is arc length on C_n. Then C_n has the representation

$$C_n: \quad z = z_n(t), \qquad 0 \leqq t \leqq 1.$$

These functions $z_n(t)$ satisfy a Lipschitz condition of constant $\mathcal{L}(C_n)$, which is less than or equal to $L+1$ by (3.12). Hence

(3.13) $$|\dot{z}_n(t)| \leqq L + 1, \qquad 0 \leqq t \leqq 1.$$

By Ascoli's theorem, we can select a subsequence of the C_n (we suppose it the whole sequence) such that the functions $z_n(t)$ converge uniformly to a limit function $z_0(t)$:

(3.14) $$\lim_{n\to\infty} z_n(t) = z_0(t) \text{ uniformly for } 0 \leqq t \leqq 1.$$

For this subsequence (3.9), (3.10), and (3.11) remain valid. The curve $z = z_0(t)$, $(0 \leqq t \leqq 1)$, we denote by C_0.

Next we define

(3.15) $$\phi_n(t) = \int_0^t F(z_n(t), \dot{z}_n(t))dt, \qquad n = 0, 1, \cdots,$$

(3.16) $$\gamma_n(t) = \int_0^t G(z_n(t), \dot{z}_n(t))dt, \qquad n = 0, 1, \cdots.$$

These functions all satisfy the same Lipschitz condition, since the integrands are bounded. So we may select a subsequence (we again denote it by $\{C_n\}$) such that $\phi_n(t)$ and $\gamma_n(t)$ tend uniformly to limit functions $\phi(t)$ and $\gamma(t)$, respectively:

(3.17) $$\lim_{n\to\infty} \phi_n(t) = \phi(t), \qquad \lim_{n\to\infty} \gamma_n(t) = \gamma(t)$$

uniformly for $0 \leqq t \leqq 1$. For this subsequence (3.9), (3.10), (3.11), and (3.14) remain valid.

Equations (3.9), (3.10), (3.15), (3.16), and (3.17) imply

$$\phi(1) = \mu_0, \qquad \gamma(1) = l. \tag{3.18}$$

From the manner of constructing the curves C_n, we notice that for every $\epsilon > 0$ all the curves C_n with large n are polygons except for their arcs which are within the ϵ-neighborhood of the boundary of S. So if $z_0(t_0)$ is a point of C_0 interior to S, there is a neighborhood $(t_0 - \delta, t_0 + \delta)$ of t_0 such that the functions $z_n(t)$, $(t_0 - \delta < t \leqq t_0 + \delta)$, represent polygonal arcs whenever n is greater than a certain n_0. (Modifications if $t_0 = 0$ or 1 are obvious.)

4. **Some lemmas.** Let us first dispose of the trivial case for which we have $L = \lim \mathcal{L}(C_n) = 0$. This is only possible if $z_1 = z_2$, and it implies that C_0 has length 0 and consists of the one point z_1. By (3.9) and (3.10) we have $l = 0 = \mu_0 \leqq \mu$. By trivial computation we obtain $\mathcal{G}(C_0) = 0 = l$, $\mathcal{J}(C_0) = 0 \leqq \mu$. Since C_0 is thus in $K[\mathcal{G} = l]$, $\mathcal{J}(C_0) \geqq \mu$; whence $\mathcal{J}(C_0) = \mu = \mu_0 = 0$ and C_0 is the curve sought. This leaves for consideration the principal case, in which

$$L = \lim \mathcal{L}(C_n) > 0. \tag{4.1}$$

Since ϕ, ϕ_0, γ, γ_0, and z_0 are all Lipschitzian functions of t, the interval $[0, 1]$ contains a set E of measure 1 such that for all t in E all the functions mentioned have derivatives and

$$\phi_0'(t) = F(z_0(t), z_0'(t)), \qquad \gamma_0'(t) = G(z_0(t), z_0'(t)). \tag{4.2}$$

It will be supposed (without loss of generality) that neither 0 nor 1 is in E. We now begin to prove a sequence of lemmas.

LEMMA 1. *If t_0 is in E, and a and b are numbers such that $aF(z, r) + bG(z, r)$ is p.q.r. at $z_0(t_0)$, then*

$$a[\phi'(t_0) - \phi_0'(t_0)] + b[\gamma'(t_0) - \gamma_0'(t_0)] \geqq 0.$$

Except for notation, this is merely a restatement of Lemma 5 of III.

LEMMA 2. *If t is in E and $z(t)$ is an ordinary point, then*

$$\phi'(t_0) = \phi_0'(t_0), \quad \gamma'(t_0) = \gamma_0'(t_0). \tag{4.3}$$

By (2.2), (a), $z_0(t_0)$ is interior to S. Therefore, as remarked at the end of §3, for all large n the arcs of C_n lying in a neighborhood of $z_0(t_0)$ are polygonal. All the hypotheses leading up to equations (8.2) and (8.3) of IV (which together are (4.3) above) are here satisfied, except that in IV the set S was the whole space. However, the proof of (8.2) and (8.3) was purely local in nature; the only reason for taking the whole space for S was to be sure that each point of

C_0 should be interior to S. So the proofs in IV are applicable without change, and the lemma is established.

LEMMA 3. *If $z_0(t_0)$ is a singular point, and t_0 belongs to E but does not belong either to $T_+(C_0)$ or to $T_-(C_0)$, then $\gamma'(t_0)=\gamma_0'(t_0)$ and $\phi'(t_0)\geqq\phi_0'(t_0)$.*

If $G(z, r)$ were not p.q.r. at $z_0(t_0)$, the point t_0 would be in $T_-(C_0)$. If it were not n.q.r. at $z_0(t_0)$, then t_0 would be in $T_+(C_0)$. So $G(z, r)$ must be both p.q.r. and n.q.r. at $z_0(t_0)$. That is, $G(z_0(t_0), r)$ is linear in the variables r. By hypothesis (2.3), $F(z, r)-\theta G(z, r)$ is p.q.r. at $z_0(t_0)$ for some θ. But since $G(z, r)$ is merely linear in the r^i, this implies that $F(z, r)$ itself is p.q.r. at $z_0(t_0)$. The application of Lemma 1 with $a=1$, $b=0$ yields $\phi'(t_0)-\phi_0'(t)\geqq 0$. Since $G(z, r)$ is both p.q.r. and n.q.r. at $z_0(t_0)$, we apply Lemma 1 with $a=0$, $b=1$ and with $a=0$, $b=-1$. This yields two inequalities which together imply $\gamma'(t_0)=\gamma_0'(t_0)$, completing the proof.

LEMMA 4. *If t_0 is in $ET_+(C_0)$, then*

$$\phi'(t_0) - \phi_0'(t_0) \geqq M(z_0(t_0))[\gamma'(t_0) - \gamma_0'(t_0)].$$

By definition of $M(z)$, there is a sequence $\{\theta_n\}$ of numbers tending to $M(z_0(t_0))$ such that for each n, $F(z, r)-\theta_n G(z, r)$ is p.q.r. at $z_0(t_0)$. By lemma 1,

$$\phi'(t_0) - \phi_0'(t_0) \geqq \theta_n[\gamma'(t_0) - \gamma_0'(t_0)].$$

Letting $n\to\infty$ establishes the desired inequality.

LEMMA 5. *If t_0 is in $ET_-(C_0)$, then*

$$\phi'(t_0) - \phi_0'(t_0) \geqq m(z_0(t_0))[\gamma'(t_0) - \gamma_0'(t_0)].$$

Choose a sequence $\{\theta_n\}$ such that $\theta_n\to m(z_0(t_0))$ and $F(z, r)-\theta_n G(z, r)$ is p.q.r. at $z_0(t_0)$ for each n. The rest of the proof is a repetition of that of Lemma 4.

LEMMA 6. *If t_0 is in E and $\gamma'(t_0)=\gamma_0'(t_0)$, then $\phi'(t_0)\geqq\phi_0'(t_0)$.*

If $z_0(t_0)$ is ordinary, this follows from Lemma 2. If $z_0(t_0)$ is singular, either it is in neither $T_+(C_0)$ nor $T_-(C_0)$, in which case $\phi'(t_0)\geqq\phi_0'(t_0)$ by Lemma 3, or it is in one (or both) of the sets $T_+(C_0)$ and $T_-(C_0)$, in which case $\phi'(t_0)-\phi_0'(t_0)\geqq 0$ by Lemma 4 or Lemma 5.

LEMMA 7. *If t_0 is in E and $\gamma'(t_0)>\gamma_0'(t_0)$, then t_0 is in $T_+(C_0)$; if t_0 is in E and $\gamma'(t_0)<\gamma_0'(t_0)$, then t_0 is in $T_-(C_0)$.*

The point $z_0(t_0)$ must be singular by Lemma 2. If t_0 is not in $T_+(C_0)$, then $G(z, r)$ is n.q.r. at $z_0(t_0)$. Applying Lemma 1 with $a=0$, $b=-1$, we obtain $\gamma'(t_0)-\gamma_0'(t_0)\leqq 0$. Hence if $\gamma'(t_0)-\gamma_0'(t_0)>0$, then t_0 is in $T_+(C_0)$. The proof of the other statement is similar.

5. **Proof of the theorem.** We now subdivide the set E into three subsets. The set T_0 will be the subset of E on which $\gamma'(t)=\gamma_0'(t)$; the set T_1 will be the subset of E on which $\gamma'(t)>\gamma_0'(t)$; and the set T_2 will be the subset of E on which $\gamma'(t)<\gamma_0'(t)$. These sets are clearly measurable, since $\gamma'(t)$ and $\gamma_0'(t)$ are measurable functions. By Lemma 7, T_1 is contained in $T_+(C_0)$ and T_2 is contained in $T_-(C_0)$.

First we shall construct a curve Γ_1: $z=\zeta_1(\tau)$, $(0\leqq\tau\leqq\tau_1)$, beginning and ending at a point $z_0(t_1)$ on C_0, and such that

$$\mathcal{G}(\Gamma_1) = \int_{T_1} [\gamma'(t) - \gamma_0'(t)]dt, \tag{5.1}$$

$$\mathcal{J}(\Gamma_1) \leqq \int_{T_1} [\phi'(t) - \phi_0'(t)]dt. \tag{5.2}$$

Whenever T_1 is empty we can take Γ_1 to be a degenerate curve consisting of a single point on C_0. Then (5.1) and (5.2) obviously hold. If T_1 is not empty, then $T_+(C_0)$ is also not empty. By hypothesis (c) there is a curve Γ^* corresponding to C_0 and having the properties there specified. For $0\leqq\tau\leqq\epsilon$ we define $\Gamma(\tau)$ to be the curve obtained by traversing Γ^* from $\zeta(0)$ to $\zeta(\tau)$ and then returning to $\zeta(0)$. Thus $\Gamma(\tau)$ is defined by the equations

$$z = \zeta(t),\ 0 \leqq t \leqq \tau; \qquad z = \zeta(2\tau - t),\ \tau < t \leqq 2\tau.$$

This is a rectifiable continuous curve beginning and ending at $\zeta(0)$, which, by hypothesis (c), is a point $z_0(t_1)$ on C_0. We calculate

$$\begin{aligned}\mathcal{G}(\Gamma(\tau)) &= \int_0^\tau G(\zeta(t), \dot\zeta(t))dt + \int_\tau^{2\tau} G(\zeta(2\tau - t), -\dot\zeta(2\tau - t))dt \\ &= \int_0^\tau [G(\zeta(t), \dot\zeta(t)) + G(\zeta(t), -\dot\zeta(t))]dt.\end{aligned} \tag{5.3}$$

Likewise

$$\mathcal{J}(\Gamma(\tau)) = \int_0^\tau [F(\zeta(t), \dot\zeta(t)) + F(\zeta(t), -\dot\zeta(t))]dt. \tag{5.4}$$

By (2.6) the integrand in (5.3) is almost everywhere positive; hence $\mathcal{G}(\Gamma(\epsilon))>0$. Let m be an integer for which

$$m\,\mathcal{G}(\Gamma(\epsilon)) > \int_{T_1} [\gamma'(t) - \gamma_0'(t)]dt.$$

Since $\mathcal{G}(\Gamma(\tau))$ is a continuous function of τ, there is a τ_0 such that

$$\mathcal{G}(\Gamma(\tau_0)) = m^{-1}\int_{T_1}[\gamma'(t) - \gamma_0'(t)]dt.$$

We now define Γ_1 to be the curve obtained by traversing $\Gamma(\tau_0)$ a total of m times. Then (5.1) holds.

Recalling that $M(C_0)$ is the greatest lower bound of $M(z_0(t))$ for all t in $T_+(C_0)$, by Lemma 4 we find that whenever t_0 is in T_1 the inequality

$$\phi'(t_0) - \phi_0'(t_0) \geqq M(C_0)[\gamma'(t_0) - \gamma_0'(t_0)]$$

holds. Hence

$$\int_{T_1}[\phi'(t) - \phi_0'(t)]dt \geqq M(C_0)\int_{T_1}[\gamma'(t) - \gamma_0'(t)]dt. \tag{5.5}$$

On the other hand, by (2.7), (5.3), and (5.4) we find

$$\begin{aligned} \mathcal{J}(\Gamma_1) &= m\int_0^{\tau_0}[F(\zeta(t), \dot\zeta(t)) + F(\zeta(t), -\dot\zeta(t))]dt \\ &\leqq mM(C_0)\int_0^{\tau_0}[G(\zeta(t), \dot\zeta(t)) + G(\zeta(t), -\dot\zeta(t))]dt \qquad (5.6)\\ &= M(C_0)\,\mathcal{G}(\Gamma_1). \end{aligned}$$

From (5.1), (5.5), and (5.6) we obtain (5.2).

Next we prove that there is a curve Γ_2 beginning and ending at a point $z_0(t_2)$ on C_0 and such that

$$\mathcal{G}(\Gamma_2) = \int_{T_2}[\gamma'(t) - \gamma_0'(t)]dt, \tag{5.7}$$

$$\mathcal{J}(\Gamma_2) \leqq \int_{T_2}[\phi'(t) - \phi_0'(t)]dt. \tag{5.8}$$

We could prove this as we did (5.1) and (5.2). But it is much simpler to observe that if we replace $G(z, r)$ by $-G(z, r)$, then $\gamma(t)$ is replaced by $-\gamma(t)$ and $M(C_0)$ by $-m(C_0)$, while hypotheses (c) and (d) are interchanged. Then (5.7) and (5.8) are merely (5.1) and (5.2) as rewritten for F and $-G$ in place of F and G.

We can now define the minimizing curve $\overline{C}$. Suppose to be specific that $t_1 \leqq t_2$. We obtain $\overline{C}$ by traversing C_0 from $t=0$ to $t=t_1$, traversing Γ_1, continuing along C_0 from $t=t_1$ to $t=t_2$, traversing Γ_2, then proceeding along C_0 from $t=t_2$ to $t=1$. We therefore have, by (3.16), (5.1), (5.7), the definition of T_0, and (3.18),

$$
\begin{aligned}
\mathcal{G}(\bar{C}) &= \mathcal{G}(C_0) + \mathcal{G}(\Gamma_1) + \mathcal{G}(\Gamma_2) \\
&= \int_{T_0} + \int_{T_1} + \int_{T_2} \gamma_0'(t)dt + \int_{T_1} [\gamma'(t) - \gamma_0'(t)]dt \\
&\quad + \int_{T_2} [\gamma'(t) - \gamma_0'(t)]dt \\
&= \int_E \gamma'(t)dt = \int_0^1 \dot{\gamma}(t)dt = \gamma(1) = l.
\end{aligned} \tag{5.9}
$$

Similarly, using (3.15), (5.2), (5.8), the definition of T_0, Lemma 6, and (3.18), we obtain

$$
\begin{aligned}
\mathcal{J}(\bar{C}) &= \mathcal{J}(C_0) + \mathcal{J}(\Gamma_1) + \mathcal{J}(\Gamma_2) \\
&\leqq \int_{T_0} + \int_{T_1} + \int_{T_2} \phi_0'(t)dt + \int_{T_1} [\phi'(t) - \phi_0'(t)]dt \\
&\quad + \int_{T_2} [\phi'(t) - \phi_0'(t)]dt \\
&\leqq \int_E \phi'(t)dt = \int_0^t \dot{\phi}(t)dt = \phi(1) = \mu_0.
\end{aligned} \tag{5.10}
$$

But by (5.9) the curve $\bar{C}$ is in $K[\mathcal{G}=l]$; so $\mathcal{J}(\bar{C}) \geqq \mu \geqq \mu_0$. This, with (5.10), implies

$$
\mathcal{J}(\bar{C}) = \mu_0 = \mu, \tag{5.11}
$$

and the proof of the theorem is complete.

Incidentally we have proved that under the hypotheses of Theorem 2 the equation $\mu_0 = \mu$ holds. It follows with little difficulty that the value of μ, considered as a function of l, is lower semicontinuous.

6. **Corollaries and examples.** Let us define $T_+^*(C)$ and $T_-^*(C)$ by deleting the words "$z(t)$ is a singular point and" in (2.3) and (2.4), and let $M^*(C)$ and $m^*(C)$ be the numbers defined by replacing $T_+(C)$, $T_-(C)$ by $T_+^*(C)$, $T_-^*(C)$, respectively, in (2.5). Then if $m(C)$ is defined, so is $m^*(C)$; and $m^*(C) \geqq m(C)$, for $T_-^*(C)$ contains $T_-(C)$. Likewise, if $M(C)$ is defined, so is $M^*(C)$; and $M^*(C) \leqq M(C)$. The following corollary is then immediately evident:

Corollary 1. *If the hypotheses of Theorem 2 hold with $m^*(C)$, $M^*(C)$ in place of $m(C)$ and $M(C)$, respectively, then the class $K[\mathcal{G}=l]$ either is empty or contains a curve for which $\mathcal{J}(C)$ assumes its least value on $K[\mathcal{G}=l]$.*

For if the hypotheses of Corollary 1 are satisfied, so are the hypotheses of Theorem 2.

In my dissertation† I established an existence theorem which overlaps considerably with Corollary 1 but neither contains it nor is contained in it. Nor does Corollary 1 cover the five existence theorems for isoperimetric problems given by Tonelli;‡ for Tonelli allows his class K to be a "complete class of total ramification," where our class K consists of the family of curves in S joining two fixed points.§ In all other respects, however, Corollary 1 contains Tonelli's theorems. Take for example Tonelli's Theorem 3 (p. 473) whose generalization to q dimensions is as follows:

Let S be bounded and closed, and let K be a complete class of curves of total ramification lying in S. Let $F(z, r)$ be p.q.r. on S, and let $G(z, r)=g(z)G(z, r)+a_\alpha(z)r^\alpha$, where $g(z)$ is nonnegative [nonpositive] on S, and through each point z_1 of S there passes an arc Γ^* on which $g(z)>g(z_1)$ $[g(z)<g(z_1)]$, provided that any continuous curve at all passes through z_1. Then $K[\mathcal{G}=l]$ either is empty or contains a curve for which $\mathcal{J}(C)$ assumes its least value on $K[\mathcal{G}=l]$.

We disregard the statements in brackets; they interchange with the unbracketed statements if G is replaced by $-G$, which replacement does not affect the hypotheses of Theorem 2 or Corollary 1. The set $T_+^*(C)$ consists of all t at which $g(z(t))>0$ and $F(z, r)$ is not linear in the r^i. The set $T_-^*(C)$ is empty; so hypothesis (d) is satisfied. For each t in $T_+^*(C)$ the function

$$F(z, r) - \theta G(z, r) = F(z, r)[1 - \theta g(z)] - \theta a_\alpha r^\alpha$$

is p.q.r. for all $\theta \leqq 1/g(z)$. Hence $M(z)=1/g(z)$. If $T_+^*(C)$ is not empty, then the greatest lower bound of $M(z(t))$ for t in $T_+^*(C)$ is at least g.l.b. $[1/g(z(t))]$. That is, $M^*(C) \geqq 1/\max g(z(t))$. Let $z_1=z(t_1)$ be a point at which $g(z(t))$ assumes its maximum (greater than zero), and let Γ^* be the curve along which $g(z) \geqq g(z_1)$. Then along Γ^* the conditions (2.6) and (2.7) are satisfied. Hypotheses (1.1) and (2.2) obviously hold. So except for the added generality of the class K, this theorem is contained in Corollary 1.

As an example covered by Theorem 2 but not by Corollary 1 or any other theorems cited, let us consider

$$F(x, y, x', y') = -(x-y)^2(x'^2 + 8y'^2)^{1/2},$$
$$G(x, y, x', y') = (x'^2 + y'^2)^{1/2},$$

† *Semi-continuity in the calculus of variations, and absolute minima for isoperimetric problems*, published in *Contributions to the Calculus of Variations*, 1930, Chicago, 1931, pp. 199–243, in particular p. 220.

‡ L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2, pp. 466–482.

§ It would, however, be quite easy to extend Corollary 1 to cover such classes K of curves. The only reason for not considering them in the first place was that the discussion of ordinary points required comparison curves other than those obtained by adding a spur like Γ_1 or Γ_2 to a given curve. In Corollary 1 the characteristic properties of ordinary points are ignored; so this need disappears.

where the range S of (x, y) is the whole plane. Hypothesis (d) of Corollary 1 is not satisfied; inequality (2.7) cannot be satisfied unless C lies entirely along the line $y=x$. However, every point (x, y) not on the line $y=x$ is an ordinary point. For, first, G is regular. Second, the matrix $\Delta(x, y; p_1, p_2; q_1, q_2)$ has the form

$$\begin{pmatrix} -(y-x)\{p_1(p_1^2+8p_2^2)^{-1/2} & -8(y-x)\{p_2(p_1^2+8p_2^2)^{-1/2} \\ \quad -q_1(q_1^2+8q_2^2)^{-1/2}\} & \quad -q_2(q_1^2+8q_2^2)^{-1/2}\} \\ p_1-q_1 & p_2-q_2 \end{pmatrix}$$

if we assume (as we may) that $p_1^2+p_2^2=q_1^2+q_2^2=1$. If a vector (p_1, p_2) is given, the vector $(q_1, q_2)=(p_1, -p_2)$ is an approach set containing (p_1, p_2), and $(-p_1, p_2)$ is in another approach set containing (p_1, p_2). It is possible (though not very easy) to show that no approach set contains any other unit vectors than these. Computing Ω_H we see that it is not zero for any of the sets except in the trivial case in which the two formally different unit vectors of the approach set coincide ($p_1=0$ or $p_2=0$). Hence every point of S with $y\neq x$ is an ordinary point.

The singular points of S are thus the points (x, x). For these the function $F(x, y, x', y')-\theta G(x, y, x', y')$ reduces to $-\theta(x'^2+y'^2)^{1/2}$, which is positive quasi-regular if and only if $\theta\leqq 0$. So $m(x, x)=-\infty$ and $M(x, x)=0$. The set $T_-(C)$ is always empty, since $G(z, r)$ is positive regular. If $T_+(C)$ is not empty, then for every t in $T_+(C)$ we have $x(t)=y(t)$ and $M(x(t), y(t))=0$. Therefore $M(C)$ is 0 whenever it is defined; that is, whenever C intersects the line $y=x$. Thus if C does not intersect the line, then $T_+(C)$ is empty; and if C intersects the line at a point (x_0, x_0), we can take Γ^* to be a segment of $y=x$ beginning at (x_0, x_0). Hypothesis (d) therefore is satisfied.

If we use the same G, but take

$$F = e^{(y+x)^2}(x'^2+8y'^2)^{1/2}$$

and let S be the whole (x, y)-plane, we find similarly that there are no singular points.

7. **A generalization.** There are several ways of strengthening Theorem 2 without great difficulty. An obvious one is as follows: If z_1, z_2, and l are given, under hypothesis (1.1) there is only a bounded subset of S which can contain points of curves C of $K[\mathcal{G}=l]$ with $\mathcal{J}(C)\leqq\mu+\epsilon$ for any given $\epsilon>0$. Let S_ϵ be this subset. We need then assume only that hypotheses (c) and (d) of Theorem 2 hold on the closure of S_ϵ.

A less trivial generalization is obtained by redefining $M(z)$ and $m(z)$ at singular points z which are interior to S. Let z be such a singular point, and let A be an approach set at z. Consider the aggregate of numbers θ for which

$$(7.1) \qquad a_\beta \mathcal{E}_{F-\theta G}(z, a_\alpha p_\alpha, p_\beta) \geqq 0$$

for all finite collections $p_1, \cdots, p_n$ of vectors of A and all sets $a_1, \cdots, a_n$ of nonnegative numbers such that $|a_\alpha p_\alpha| \neq 0$. We make the following definitions:

(7.2) *$M_1(z, A)$ is the least upper bound of all numbers θ such that* (7.1) *holds, and $m_1(z, A)$ is their greatest lower bound.*

(7.3) *$M_1(z)$ is the greatest lower bound of $M_1(z, A)$ for all approach sets A at z, and $m_1(z)$ is the least upper bound of $m_1(z, A)$ for all approach sets A at z.*

Under hypothesis (2.2) such numbers θ exist. For if θ serves in (2.2), then

$$\mathcal{E}_{F-\theta G}(z, p, r) \geqq 0$$

for all $p \neq 0$ and all r, and (7.1) follows at once. This argument shows moreover that every θ which serves in (2.2) serves in (7.1), no matter which approach set A we use. Hence if A is any approach set at z, $M_1(z, A) \geqq M(z)$ and $m_1(z, A) \leqq m(z)$; so by (7.3)

$$(7.4) \qquad M_1(z) \geqq M(z), \qquad m_1(z) \leqq m(z).$$

Our theorem is given as follows:

THEOREM 3. *At all singular points z interior to S let $M(z)$, $m(z)$ be redefined to mean $M_1(z)$, $m_1(z)$, respectively. Then with this new meaning of $m(z)$ and $M(z)$ Theorem 2 remains valid.*

The numbers $M(z)$, $m(z)$ entered the proof of Theorem 2 by way of Lemmas 4 and 5. Therefore we need only establish Lemmas 4 and 5 with M_1, m_1 in place of M, m, respectively. Suppose then that $z_0(t_0)$ is a singular point interior to S. By Lemma 5 of IV there is an approach set A and a subsequence $\{z_m(t)\}$ with the properties there specified. (We disregard case (i) of that Lemma, for then the proof that $\phi'(t_0) = \phi_0'(t_0)$ and $\gamma'(t_0) = \gamma_0'(t_0)$ goes through as before.) By the definition of the $\mathcal{E}$-function, inequality (7.1) can be written

$$(7.5) \qquad a_\alpha[F(z_0, p_\alpha) - \theta G_\alpha(z_0, p_\alpha)] \geqq F(z_0, a_\alpha p_\alpha) - \theta G(z_0, a_\alpha p_\alpha),$$

where we have written z_0 for $z_0(t_0)$. Let R be the (convex) set consisting of all nonzero vectors r which can be written in the form $a_1 p_1 + \cdots + a_n p_n$, where each p_i is in A and $a_i \geqq 0$. Each r in R can be written in one or many ways as a sum $a_\alpha p^\alpha$. The lower bound of the left member of (7.5), for all such ways of writing r, is known to be a convex function $\overline{H}(z_0, r)$ on R. By (7.5),

$$\overline{H}(z_0, r) \geqq F(z_0, r) - \theta G(z_0, r), \qquad r \text{ in } R.$$

From this (and the differentiability of F and G) there is, for each r_0 in R, a linear function $l_\alpha r^\alpha$ such that

$$F(z_0, r_0) - \theta G(z_0, r_0) \leqq l_\alpha r_0^\alpha, \tag{7.6}$$

$$\overline{H}(z_0, r) \geqq l_\alpha r^\alpha \text{ for all } r \text{ in } R. \tag{7.7}$$

In particular, $F(z_0, p) - \theta G(z_0, p) \geqq \overline{H}(z_0, p)$ for all p in A, as we find by taking $n=1$, $p_1=p$, $a_1=1$, in (7.5). So by (7.7)

$$F(z_0, p) - \theta G(z_0, p) \geqq l_\alpha p^\alpha, \qquad p \text{ in } A. \tag{7.8}$$

Let us denote the closed γ-neighborhoods of z_0, A, R by $(z_0)_\gamma$, $(A)_\gamma$, $(R)_\gamma$, respectively. For each $\gamma>0$, if m is large and δ small, the point $z_m(t)$ is $(z_0)_\gamma$ and $z_m'(t)$ is in $(A)_\gamma$ for $t_0-\delta\leqq t\leqq t_0+\delta$. A fortiori, $z_m'(t)$ is in the closed convex set $(R)_\gamma$. By Jensen's inequality, if $t_0-\delta\leqq t<t+h\leqq t_0+\delta$, then

$$\frac{1}{h}\int_t^{t+h} \dot{z}_m(t)dt \equiv [z_m(t+h) - z_m(t)]/h$$

is in $(R)_\gamma$. Let $m\to\infty$; the vector $[z_0(t+h)-z_0(t)]/h$ is in $(R)_\gamma$. Let $h\to 0$; the vector $z_0'(t)$, if defined, is in $(R)_\gamma$. By use of fairly obvious estimates, we find by (7.6) and (7.8) that for all sufficiently small positive numbers γ

$$F(z_0(t), z_0'(t)) - \theta G(z_0(t), z_0'(t)) \leqq l_\alpha z_0^{\alpha\prime}(t) + \epsilon, \tag{7.9}$$

$$F(z_m(t), z_m'(t)) - \theta G(z_m(t), z_m'(t)) \geqq l_\alpha z_m^{\alpha\prime}(t) - \epsilon, \qquad t_0 - \delta \leqq t < t_0 + \delta, \ m \text{ large}. \tag{7.10}$$

Integrating from t_0 to t_0+h yields

$$\phi_0(t_0+h) - \phi_0(t_0) - \theta[\gamma_0(t_0+h) - \gamma_0(t_0)] \leqq l_\alpha[z_0^\alpha(t_0+h) - z_0^\alpha(t_0)] + \epsilon h,$$

$$\phi_m(t_0+h) - \phi_m(t_0) - \theta[\gamma_m(t_0+h) - \gamma_m(t_0)] \geqq l_\alpha[z_m^\alpha(t_0+h) - z_m^\alpha(t_0)] - \epsilon h.$$

If we let $m\to\infty$, divide by h, and let $h\to\infty$, we get

$$\phi_0'(t_0) - \theta\gamma_0'(t_0) \leqq l_\alpha z_0^{\alpha\prime}(t_0) + \epsilon, \qquad \phi'(t_0) - \theta\gamma'(t_0) \geqq l_\alpha z_0^{\alpha\prime}(t_0) - \epsilon.$$

Since ϵ is arbitrary,

$$\phi'(t_0) - \phi_0'(t_0) \geqq \theta[\gamma'(t_0) - \gamma_0'(t_0)].$$

If we let θ run through a sequence of values approaching $M_1(z_0(t_0), A)$, we find

$$\phi'(t_0) - \phi_0'(t_0) \geqq M_1(z_0(t_0), A)[\gamma'(t_0) - \gamma_0'(t_0)].$$

This holds for all $M_1(z_0(t_0), A)$; so it holds for their greatest lower bound $M_1(z_0(t_0))$. The generalization of Lemma 4 is therefore established. Lemma 5 can be discussed similarly; or we can obtain the result from the proof above by replacing G by $-G$.

From the definitions it is evident that any alteration in the definitions of $M_1(z, A)$, $m_1(z, A)$, $M_1(z)$, and $m_1(z)$ which enables us to discard vectors p from approach sets A at z or enables us to disregard entire approach sets A either leaves these numbers unchanged or improves them; that is, if $M_1(z, A)$ and $M_1(z)$ are altered by the change of definition they are increased, and if $m_1(z, A)$ and $m_1(z)$ are altered they are decreased. In §8 we shall establish a criterion which will permit us to ignore certain types of approach sets. Here we establish two simpler criteria. Suppose that z is a singular point interior to S, at which $G(z, r)$ is quasi-regular normal. If A is an approach set containing only a finite number of unit vectors $p_1, \cdots, p_k$, and if these can be so ordered that $\Omega_H(z, p_i, p_j) < 0$ if $i < j$, then A can be disregarded in defining $m_1(z)$ and $M_1(z)$. For if, in Lemma 5 of IV, the set A can be so chosen as to have these properties, all the proof leading up to equations (8.2) and (8.3) of IV remains valid without change, and we obtain $\gamma'(t_0) = \gamma_0'(t)$ and $\phi'(t_0) = \phi_0'(t_0)$.

Retaining the assumption concerning $G(z, r)$, let us suppose that A is an approach set containing a finite number of unit vectors. If there is a unit vector p_1 in A such that $\Omega_H(z, p_1, p) < 0$ for all unit vectors $p \neq p_1$ in A, we say that p_1 is the first vector in A. If there is also a unit vector p_2 in A such that $\Omega_H(z, p_2, p) < 0$ for all unit vectors p in A except p_1 and p_2, then p_2 is the second vector in A; and so on. If there is a p_l in A such that $\Omega_H(z, p, p_l) < 0$ for all unit vectors $p = p_l$ in A, we say that p_l is the last vector in t; and so on for $p_{l-1}, p_{l-2}, \cdots$. Unless z is an ordinary point, there may remain some unit vectors not thus classified. These and their multiples we call the non-ordered nucleus of A. In defining $M_1(z, A)$ and $m_1(z, A)$ we can discard from A all vectors not belonging to the non-ordered nucleus. The details of proof I shall omit.

8. **$\mathcal{E}$-Admissibility.** In §8 of III we introduced a concept called $\mathcal{E}$-admissibility, and showed that we could restrict our attention to those approach sets which were $\mathcal{E}$-admissible. The set A was $\mathcal{E}$-admissible if $\mathcal{E}(z, p_0, p) \geqq 0$ for all p_0 in A and all p. If we wish to define an analogous notion for isoperimetric problems, we must be guided by the way in which the Weierstrass condition is stated for those problems. The Weierstrass condition is to the effect that $\mathcal{E}_H(z_0(t), z_0'(t), p) \geqq 0$ if $z = z_0(t)$ is the minimizing curve and $H = F - \lambda G$. This suggests the following definition:

(8.1) *Let A be an approach set at z, and let $H(z, r)$ be the function $F(z, r) - \lambda(z, A)G(z, r)$. Then the set A is $\mathcal{E}$-admissible if*

$$\mathcal{E}_H(z, p_0, p) \geqq 0$$

for all p_0 in A and all p.

In complete analogy with III, we can prove the following theorem:

THEOREM 4. *If in the definition of ordinary point we replace the words* (*in* (2.1), (c)) "*every approach set*" *by* "*every* $\mathcal{E}$*-admissible approach set,*" *Theorems* 2 *and* 3 *remain valid.*

I have been unable to find any proof of this theorem which is not extremely long and involved. Therefore I shall here content myself with a sketch of a proof; the reader will probably be able to furnish the omitted details if he is interested.

Let μ_0 be (as before) the least number which is the limit of $\mathcal{J}(\Pi_n)$ for a sequence of polygons Π_n of K such that $G(\Pi_n)\rightarrow l$. We may suppose that Π_n has the usual minimizing property with respect to curves having not more vertices than Π_n has. We thus come to Lemma 2 and must establish that lemma. Suppose the contrary, that one of the equations (4.3) fails. In particular, we suppose that the second one fails. In the proof of Lemma 1 an approach set A entered, via Lemma 5 of IV. If this approach set is $\mathcal{E}$-admissible, the whole argument leading to equations (4.3) is valid without alteration, and (4.3) holds. This is a contradiction. It remains to consider the possibility that A is not $\mathcal{E}$-admissible and show that this leads to a contradiction.

If A is not $\mathcal{E}$-admissible, there is a vector p_1 such that $\mathcal{E}_H(z, p_0, p_1)<0$ for some (hence for all) p_0 in A. Choose a small interval $[t-\delta, t_0+\delta]$. We treat the two subintervals $[t_0-\delta, t_0]$ and $[t_0, t_0+\delta]$ differently.

If, as usual, we write $H(z, r)=F(z, r)-\lambda(z_0(t_0), A)G(z, r)$, we may assume $H_i(z_0(t_0), p)=0$ for all p in A; for this may be brought about by adding the linear function $-H_\alpha(z_0(t_0), p_0)r^\alpha$ to $H(z, r)$, where p_0 is in A. Let $\zeta_m(t, 0)$ be the linear function for which $\zeta_m(t_0, 0)=z_m(t_0)$ and $\zeta_m(t_0+\delta, 0)=z_m(t_0+\delta)$, and define

$$\zeta_m(t, \sigma) = \zeta_m(t, 0) + \sigma[z_m(t) - \zeta_m(t, 0)], \qquad t_0 \leqq t \leqq t_0 + \delta.$$

Thus $\zeta_m(t, 1)\equiv z_m(t)$. To simplify the situation we shall ignore the dependence of $F(z, r)$ and $G(z, r)$ on z. It is easy to verify that all the integrals over $[t_0-\delta, t_0]$ and $[t_0, t_0+\delta]$ are thereby changed by at most $\theta(m, \delta)\delta$, where $\theta(m, \delta)$ tends to zero as $m\rightarrow\infty$ and $\delta\rightarrow 0$. Accordingly, we write $G(r)$ for $G(z, r)$, and so on.

The integral

$$I_m(\sigma) = \int_{t_0}^{t_0+\delta} G(\dot\zeta_m(t, \sigma))dt = \int_{t_0}^{t_0+\delta} G(\dot\zeta_m(t, 0) + \sigma[\dot z_m(t) - \dot\zeta_m(t, 0)])dt$$

is a convex function of σ, since $G(r)$ is convex. If we write $\gamma'(t_0)-\gamma_0'(t_0)=3\kappa$, then for all large m and small δ we have

$$[\gamma_m(t_0 + \delta) - \gamma_m(t_0)] - [\gamma_0(t_0 + \delta) - \gamma_0(t_0)] > 2\kappa\delta.$$

Since $G(r)$ is independent of z and is positive quasi-regular, the line segment $z = \zeta_m(t, 0)$ furnishes an absolute minimum for $\mathcal{G}(C)$ in the class of curves joining its ends. Hence

$$\lim_{m\to\infty} \int_{t_0}^{t_0+\delta} G(\dot{\zeta}_m)dt \leqq \int_{t_0}^{t_0+\delta} G(\dot{z}_0)dt.$$

With the preceding inequality, this shows that for large m and small δ

$$(8.2) \qquad I_m(1) = \gamma_m(t_0 + \delta) - \gamma_m(t_0) \geqq \int_{t_0}^{t_0+\delta} G(\dot{\zeta}_m)dt + \kappa\delta = I_m(0) + \kappa\delta.$$

From the convexity of $I_m(\sigma)$ we find that

$$(8.3) \qquad I_m(1 - \sigma) - I_m(1) < -\sigma\kappa\delta \quad \text{for} \quad 0 \leqq \sigma \leqq 1.$$

Now we consider the interval $(t_0 - \delta, t_0)$. On this we use the construction of §8 of III. We thereby replace the arc $z = z_m(t)$, $(t_0 - \delta \leqq t \leqq t_0)$, of Π_m by an arc $z = z_m(t, \epsilon)$ with the same ends and having

$$(8.4) \qquad \int_{t_0-\delta}^{t_0} H(\dot{z}_m(t, \epsilon))dt - \int_{t_0-\delta}^{t_0} H(\dot{z}_m(t))dt < -2\gamma\delta\epsilon, \qquad \gamma > 0,$$

for all m. Because of the convexity of $G(r)$ the integral of G is increased, but it is easy to estimate that the increase is less than $K\delta\epsilon$, K a constant.

Choose ϵ small enough so that $K\delta\epsilon < \kappa\delta$. Then by (8.3) there is a σ such that $0 \leqq \sigma \leqq 1$ and

$$(8.5) \qquad I_m(1 - \sigma) - I_m(1) = \int_{t_0-\delta}^{t_0} G(z_m'(t))dt - \int_{t_0-\delta}^{t_0} G(z_m'(t, \epsilon))dt.$$

That is, if we let $\bar{z}_m(t)$ be $z_m(t, \epsilon)$ on $[t_0 - \delta, t_0)$ and $\zeta_m(t, \sigma)$ on $[t_0, t_0 + \delta]$, then

$$(8.6) \qquad \int_{t_0-\delta}^{t_0+\delta} G(\bar{z}_m')dt = \int_{t_0-\delta}^{t_0+\delta} G(z_m')dt.$$

The right member of equation (8.5) has a value between $-K\delta\epsilon$ and zero; so by inequality (8.3) we conclude that

$$(8.7) \qquad 0 < \sigma < K\epsilon/\kappa.$$

By reducing ϵ if necessary, we can ensure that σ is less than $1/12$.

Since A is an approach set on which H vanishes identically, all first-order partial derivatives of H also vanish on A. Therefore there is a positive number λ such that

$$(8.8) \qquad H_\alpha(r)H_\alpha(r) < [\gamma\kappa/3KL]^2$$

if $L/2 \leqq |r| \leqq 3L/2$ and r is in the 2λ-neighborhood of A. If m is large, the inequality

$$3L/4 < |z_m'(t)| < 5L/4 \tag{8.9}$$

holds; and moreover $z_m'(t)$ is in the λ-neighborhood of A if $|t-t_0|<\delta$, provided that δ is small enough. By definition of ζ_m we find that

$$\begin{aligned} |\zeta_m'(t, 1-\sigma) - z_m'(t)| &= \sigma|z_m'(t) - \zeta_m'(t, 0)| \\ &< \sigma 3L < L/4. \end{aligned} \tag{8.10}$$

By using the theorem of the mean, together with (8.7) and (8.8), whose use is permitted by (8.9) and (8.10), we obtain

$$|H(\zeta_m'(t, 1-\sigma)) - H(z_m'(t))| < \sigma\gamma\kappa/K < \gamma\epsilon. \tag{8.11}$$

Recalling the definition of $\bar{z}_m(t)$; inequalities (8.4) and (8.11) (integrated from t_0 to $t_0+\delta$), we obtain

$$\begin{aligned} \int_{t_0-\delta}^{t_0+\delta} F(\bar{z}_m')dt - \int_{t_0-\delta}^{t_0+\delta} F(z_m')dt &= \int_{t_0-\delta}^{t_0+\delta} H(\bar{z}_m')dt - \int_{t_0-\delta}^{t_0+\delta} H(z_m')dt \\ &< -\gamma\delta\epsilon. \end{aligned} \tag{8.12}$$

Extending $\bar{z}_m(t)$ by setting it equal to $z_m(t)$ for $0 \leqq t < t_0-\delta$ and $t_0+\delta < t \leqq 1$, we obtain a polygon $\bar{\Pi}_m$ such that

$$\mathcal{G}(\bar{\Pi}_m) = \mathcal{G}(\Pi_m), \qquad \mathcal{J}(\bar{\Pi}_m) < \mathcal{J}(\Pi_m) - \gamma\delta\epsilon.$$

But then $\limsup \mathcal{J}(\Pi_m) \leqq \mu_0 - \gamma\delta\epsilon$, contrary to the definition of μ_0, and the desired contradiction has been reached.

We thus see that $\gamma'(t_0) = \gamma_0'(t_0)$ for almost all t_0 such that $z_0(t_0)$ is interior to S. If $t_1 \leqq t \leqq t_2$ defines an interior arc of $z = z_0(t)$, this proves that

$$\lim_{n\to\infty} \int_{t_1}^{t_2} G(z_n, \dot{z}_n)dt = \int_{t_1}^{t_2} G(z_0, \dot{z}_0)dt.$$

But G is quasi-regular normal; so by a known theorem this implies

$$\lim_{n\to\infty} \int_{t_1}^{t_2} F(z_m, \dot{z}_m)dt = \int_{t_1}^{t_2} F(z_0, \dot{z}_0)dt;$$

that is, $\phi(t_2) - \phi(t_1) = \phi_0(t_2) - \phi_0(t_1)$. Hence $\phi'(t_0) = \phi_0'(t_0)$ for all points $z_0(t_0)$ interior to S, and equations (4.3) are established.

Besides the added generality, Theorem 4 offers another advantage. The search for $\mathcal{E}$-admissible approach sets may be easier than the determination of all approach sets. For one thing, if A is an approach set at z, and there is a z' such that

$$(8.13)\qquad H(z, z'; \lambda(z, A)) + H(z, -z'; \lambda(z, A)) < 0,$$

then A is not $\mathcal{E}$-admissible, as we see if we rewrite (8.13) in the form

$$\mathcal{E}_H(z, p, z') + \mathcal{E}_H(z, p, -z') < 0.$$

If F and G belong to the important special class of integrands such that

$$F(z, -z') = F(z, z'), \qquad G(z, -z') = G(z, z'),$$

then in order that the approach set A at z be $\mathcal{E}$-admissible it is necessary that $H(z, z'; \lambda(z, A))$ be nonnegative. The example of §6 is of this type. More generally, let

$$F = \phi(x, y)(x'^2 + a^2y'^2)^{1/2}, \qquad G = \psi(x, y)(x'^2 + y'^2)^{1/2},$$

where $a > 1$ and $\psi > 0$. Here

$$H = \phi(x'^2 + a^2y'^2)^{1/2} - \lambda\psi(x'^2 + y'^2)^{1/2},$$

and in order that this be nonnegative we must have

$$(8.14)\qquad \lambda \leqq a\phi/\psi.$$

Suppose to be specific that $\phi \leqq 0$. If the equality holds in (8.14), then $H(x, y, 0, y'; \lambda) = 0$ for all y', and $(0, 1)$ and $(0, -1)$ are in an $\mathcal{E}$-admissible approach set. No other unit vectors are in this set unless $\phi = 0$. If $\lambda < a\phi/\psi$ then H is positive. The graph of $H = 1$ is either convex (if $-\lambda$ is large) or dumb-bell shaped, with its narrowest section along the x'-axis. It is then geometrically evident that the only $\mathcal{E}$-admissible approach sets are those containing only two unit vectors, (p, q) and $(p, -q)$. This applies, in particular, to the example of §6.

Again, let $F = -e^y(x'^2 + 4y'^2)^{1/2}$, $G = (x'^2 + y'^2)^{1/2}$. As we have just seen, the only $\mathcal{E}$-admissible approach sets contain at most two unit vectors, (p, q) and $(p, -q)$. Suppose $q > 0$; then $\Omega_H(x, y, p, q, p, -q) = -e^y\{2q(p^2 + 4q^2)^{1/2}\}$. So (2.1), (c) holds for the $\mathcal{E}$-admissible approach sets A, and Theorem 4 applies. But Theorems 1 and 2 do not apply. For if (p, q) is a unit vector, then $(-p, q)$ is in an approach set with (p, q), and $\Omega_H(x, y, p, q, -p, q) = 0$. Hence (2.1) is not satisfied for all approach sets A, and no point is ordinary as defined in (2.1).

UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VA.